Name (IN CAPITALS): **Version #1**

Instructor and Class Time: Isaac Newton

Math 10550 Exam 3
Nov. 16, 2023.

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(•)	(b)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice ________

11. ________

12. ________

13. ________

14. ________

Total ________

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2	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice ________________

11. ________________

12. ________________

13. ________________

14. ________________

Total ________________

Multiple Choice

1.(6pts) Compute $\lim_{x\to-\infty} \dfrac{\sqrt{4x^2+2x+3}}{3x-1}$.

Solution: $\lim_{x\to-\infty} \dfrac{\sqrt{4x^2+2x+3}}{3x-1} = \lim_{x\to-\infty} \dfrac{\sqrt{4x^2+2x+3}}{3x-1} \cdot \dfrac{\sqrt{\frac{1}{x^2}}}{\sqrt{\frac{1}{x^2}}}$

$= \lim_{x\to-\infty} \dfrac{\sqrt{4x^2+2x+3}}{3x-1} \cdot \dfrac{\sqrt{\frac{1}{x^2}}}{-\frac{1}{x}}$ (because when $x<0, -\frac{1}{x}>0$ and $(-\frac{1}{x})^2 = \frac{1}{x^2}$).

$= \lim_{x\to-\infty} \dfrac{\sqrt{4+\frac{2}{x}+\frac{3}{x^2}}}{\frac{1}{x}-3} = \dfrac{\sqrt{4+0+0}}{0-3} = -\dfrac{2}{3}$.

(a) $-2/3$ (b) $2/3$ (c) $1/3$ (d) $-1/3$ (e) 1

2.(6pts) The equation of the slant asymptote of the curve

$$y = \frac{2x^3-8x+1}{x^2-2x}$$

is:

Solution:

$$\begin{array}{r|l} & 2x+4 \\ \hline x^2-2x) & 2x^3 \qquad -8x+1 \\ & -2x^3+4x^2 \\ \hline & 4x^2-8x \\ & -4x^2+8x \\ \hline & 1 \end{array}$$

(a) $y=2x+4$

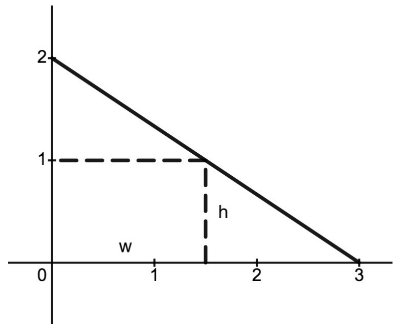
(b) $y=x-4$

(c) $y=2x-4$

(d) $y=2x$

(e) $y=x-1$

3.(6pts) Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle, with sides of length 2, 3 and $\sqrt{13}$, shown below.



Solution: Note that the hypotenuse for this triangle is just a section of the line $y = -\frac{2}{3}x + 2$. So for a rectangle of width w that is inscribed in our triangle, the height of this rectangle must be $h = -\frac{2}{3}w + 2$. The area of an arbitrary inscribed rectangle in terms of it's width w must then be $A(w) := w(-\frac{2}{3}w + 2) = -\frac{2}{3}w^2 + 2w$. To maximize the area of our inscribed rectangle, we take the derivative $A'(w) = -\frac{4}{3}w + 2$ and solve for the value of w such that $A'(w) = 0$. This happens when $w = 2(\frac{3}{4}) = \frac{3}{2}$.

(a) $w = \dfrac{3}{2},\quad h = 1$

(b) $w = \dfrac{1}{2},\ h = \dfrac{5}{3}$

(c) $w = \dfrac{6}{5},\quad h = \dfrac{6}{5}$

(d) $w = 1,\ h = \dfrac{1}{3}$

(e) $w = \dfrac{1}{3},\quad h = \dfrac{16}{9}$

4.(6pts) The equation $x^5 + 2x^3 + x - 1 = 0$ has one solution between 0 and 1. Find the result of one iteration of Newton's method applied to this equation with 1 as the starting point (i.e. find x_2 using Newton's method applied to the equation with $x_1 = 1$).

Solution: Recall the formula for Newtons method is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Therefore $x_2 = x_1 - \frac{f(1)}{f'(1)}$ and we are given $x_1 = 1$ so $f(x_1) = f(1) = 1 + 2 + 1 - 1 = 3$ and $f' = 5x^4 + 6x^2 + 1$ so $f'(x_1) = f'(1) = 5 + 6 + 1 = 12$. Putting all this together we get $x_2 = 1 - \frac{3}{12} = \frac{9}{12} = \frac{3}{4}$.

(a) $\dfrac{3}{4}$ (b) $-\dfrac{1}{3}$ (c) $\dfrac{5}{3}$ (d) $\dfrac{5}{4}$ (e) $\dfrac{11}{12}$

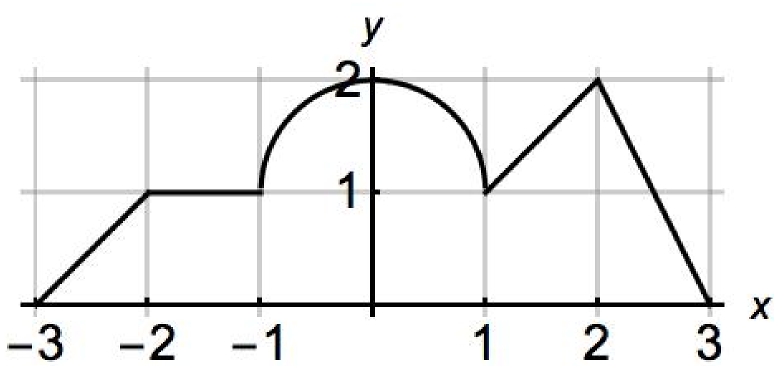
5.(6pts) If $\int_2^3 f(x)\,dx = 1$, $\int_3^5 f(x)\,dx = 2$ and $\int_2^1 f(x)\,dx = 4$, find $\int_1^5 f(x)\,dx$.

Solution: Recall that in general we have $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ and $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$. Therefore

$$\int_1^5 f(x)\,dx = \underbrace{\int_1^2 f(x)\,dx}_{-4} + \underbrace{\int_2^3 f(x)\,dx}_{1} + \underbrace{\int_3^5 f(x)\,dx}_{2} = -1$$

(a) -1 (b) 1 (c) 7 (d) 3 (e) -2

6.(6pts) The graph of $g(x)$ shown below consists of four straight lines and a semicircle. Use it to calculate the integral $\int_{-3}^{3} g(x)dx$.



Solution: $\int_3^{-3} g(x)\,dx = \int_{-1}^{-3} g(x)\,dx + \int_{-1}^{1} g(x)\,dx + \int_1^2 g(x)\,dx \int_2^3 g(x)\,dx$

The first integral is equivalent to the area of a trapezoid with base lengths 1 and 2 of height 1 (as is the third integral). The second integral is equivalent to the area of a semicircle of radius 1 plus the area of a 2×1 rectangle. The fourth integral is equivalent to the area of a 1×2 right triangle.

Thus $\int_3^{-3} g(x)\,dx = \frac{1+2}{2}(1) + \frac{1}{2}\pi(1)^2 + 2(1) + \frac{1+2}{2}(1) + \frac{1}{2}(1)(2) = \frac{\pi + 4 + 3 + 3 + 2}{2}$
$= \frac{\pi + 12}{2}$.

(a) $\frac{\pi + 12}{2}$ (b) $\frac{13}{2}$ (c) $\frac{22}{3}$ (d) $2(\pi + 3)$ (e) $\pi + 6$

7.(6pts) Let $f(x) = \int_0^{x^2} \sqrt{1+t^2}\, dt$. Find $f'(x)$.

Solution: Notice that $\sqrt{1+t^2}$ is a continuous function on all real numbers t so if we let

$$h(x) = \int_0^{x} \sqrt{1+t^2}\, dt \implies h(x^2) = f(x) = \int_0^{x^2} \sqrt{1+t^2}\, dt$$

so by the chain rule $f'(x) = 2xh'(x^2)$ but by the Fundamental Theorem of Calculus we have that $h'(x) = \sqrt{1+x^2}$ and hence $f'(x) = 2xh'(x^2) = 2x\sqrt{1+(x^2)^2} = 2x\sqrt{1+x^4}$

(a) $2x\sqrt{1+x^4}$ (b) $\sqrt{1+x^4}$ (c) $2x\sqrt{1+x^2}$ (d) $\sqrt{1+4x^2}$ (e) $\dfrac{x}{\sqrt{1+x^4}}$

8.(6pts) Evaluate

$$\int_0^{\frac{\pi}{4}} \cos t - \sin t\, dt$$

Solution: Recall that in general we have $\int_a^b f(x) - g(x)\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$. Therefore we have that

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \cos t - \sin t\, dt &= \int_0^{\frac{\pi}{4}} \cos t\, dt - \int_0^{\frac{\pi}{4}} \sin t\, dt \\ &= \sin t \,\Big|_0^{\frac{\pi}{4}} - (-\cos t)\,\Big|_0^{\frac{\pi}{4}} \\ &= \frac{\sqrt{2}}{2} - 0 - \left(\frac{-\sqrt{2}}{2} - (-1)\right) = \sqrt{2} - 1\end{aligned}$$

(a) $\sqrt{2}-1$ (b) $\sqrt{2}+1$ (c) $\dfrac{\sqrt{3}-1}{2}$ (d) $\dfrac{\sqrt{3}+1}{2}$ (e) 0

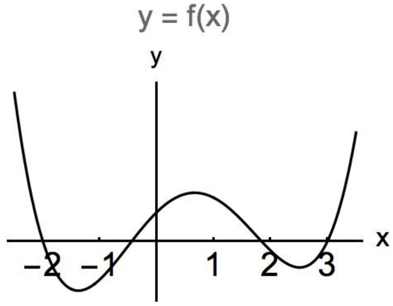
9.(6pts) Calculate the following definite integral

$$\int_1^2 \frac{2x+3}{(x^2+3x+2)^2}\,dx$$

Solution: We will compute the integral via $u-$substitution. Let $u = x^2+3x+2$ so our integral now becomes $\int_1^2 \frac{2x+3}{u^2}\,dx$. Then $du = (2x+3)dx$ which we use to replace our integral to get $\int_1^2 \frac{1}{u^2}\,du$. Note that our bounds on our integral were bounds for x so we need to replace them with the correct bounds for u. At $x = 1$ we have that $u = (1)^2 + 3(1) + 2 = 6$ and at $x = 2$ we have that $u = (2)^2 + 3(2) + 2 = 12$. Thus our integral becomes $\int_6^{12} \frac{1}{u^2}\,du$ or equivalently $\int_6^{12} u^{-2}\,du$. Notice that now we can apply our integration rules to get $\int_6^{12} u^{-2}\,du = -u^{-1}\mid_6^{12} = (\frac{-1}{12}) - (\frac{-1}{6}) = \frac{-1}{12} + \frac{2}{12} = \frac{1}{12}$.

(a) $\frac{1}{12}$ (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ (d) $-\frac{3}{2}$ (e) $\frac{7}{24}$

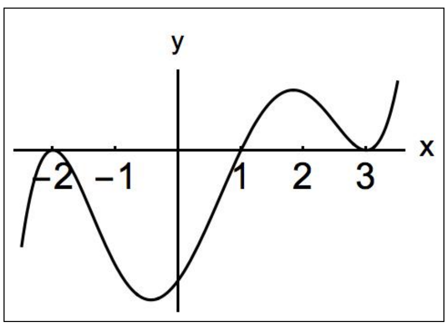
10.(6pts) The graph of $f(x)$ is shown below:



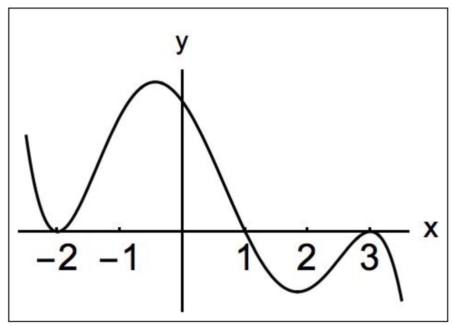
Which of the following gives the graph of an antiderivative for the function $f(x)$?

Solution: An antiderivative F of a function f is a function such that $F'(x) = f(x)$. Therefore, changes in sign on the graph of $f = F'$ should correspond to relative minima and maxima on the graph of F. We see from the graph of f that $F'(x)$ changes from positive to negative when $x = -2$ and from negative to positive when $x = 3$. Therefore F must have a relative maximum at $x = -2$ and a relative minimum at $x = 3$. Graph (a) is the only graph fulfilling this criterion.

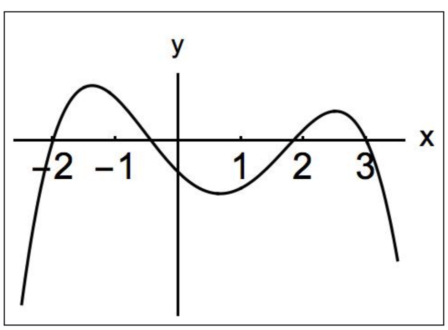
(a)



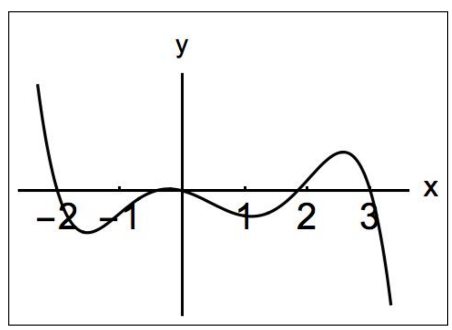
(b)



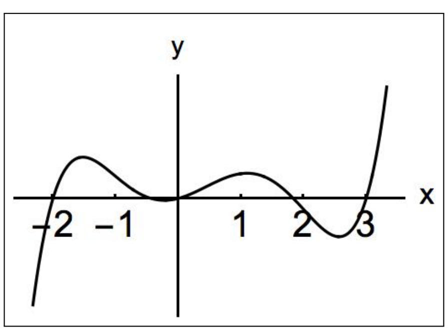
(c)



(d)



(e)

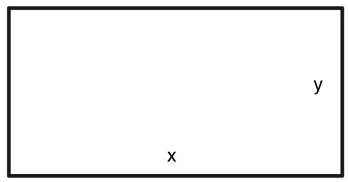


Partial Credit

Make sure you **justify all of your answers** to receive full credit for questions 11-13.

11.(12pts) A farmer wants to build a rectangular enclosure (shown below) with the front side made of material costing \$20 per meter and with the other three sides (back, left, right) of material costing \$10 per meter. The farmer has \$360 to spend. What dimensions (x and y) ensure the maximum area for the enclosure?

Show all work, and *make sure you justify that your answer is a maximum.*



Solution: Let's first define a cost function C that will take our dimensions x, y as inputs (in meters) and output the cost of material (in dollars). Let $C(x, y) = 20 \cdot x + 10 \cdot (2y + x)$. The farmer wants to spend \$360 so we have $360 = 20x + 10(2y + x) = 30x + 20y$ which gives us a relation between x and y. Namely, $\frac{360-20y}{30} = x$ or equivalently $\frac{360-30x}{20} = y$. Now that we have utilized the given information, lets turn to maximizing the area of the enclosure. Recall that the area of the enclosure is given by $A = xy$. Inputting our relation from above to get A as a function of a single variable we have that $A = x\left(\frac{360-30x}{20}\right) = \frac{360x-30x^2}{20} = 18x - \frac{3}{2}x^2$. Then $A' = 18 - 3x$ and therefore the critical point of A is at $x = 6$. Note that in order to claim A reaches a maximum at $x = 6$ we need to verify via the first derivative test which gives us that $A' > 0$ for $0 < x < 6$ and $A' < 0$ for $6 < x < \infty$ so A reaches a maximum at $\underline{x = 6}$ and $\underline{y = \frac{360-30x}{20} = 9}$.

$x =$ ______________ $y =$ ______________

10. Initials: ____________

12.(14pts) (a) Evaluate the definite integral $\int_0^2 \frac{x^2}{3} dx$ by using right endpoints and the **limit definition** of the definite integral.

Hint: $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1)$.

Solution:

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 0 + i(\frac{2}{n}) = \frac{2i}{n}$$

$$\int_0^2 \frac{x^2}{3} dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{(\frac{2i}{n})^2}{3} \frac{2}{n} = \lim_{n\to\infty} (\frac{8}{3n^3}) \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} (\frac{8}{3n^3}) \frac{(n)(n+1)(2n+1)}{6} = \lim_{n\to\infty} \frac{8(n+1)(2n+1)}{18n^2}$$

$$= \lim_{n\to\infty} \frac{8n^2 + 12n + 4}{9n^2} = \lim_{n\to\infty} \frac{8}{9} + \frac{4}{3n} + \frac{4}{9n^2}$$

$$= \frac{8}{9}$$

(b) Verify your answer by using the Fundamental Theorem of Calculus to evaluate $\int_0^2 \frac{x^2}{3} dx = \int_0^2 \frac{x^2}{3} dx$.

Solution: $\int_0^2 \frac{x^2}{3} dx = \text{(FTC PART II)} \left. \frac{x^3}{9} \right|_0^2 = \frac{8}{9}$

13.(12pts) The velocity of a particle (in meters per second) is given by

$$v(t) = t^2 - 5t + 4.$$

(a) Express the **displacement** of the particle over the interval $[0, 2]$ as a definite integral.

Solution: The displacement of the paritcle is just the final position minus the initial position. By FTC part 2, as the derivative of the position function is the velocity function, the displacement $s(2) - s(0)$ can be given by $\int_0^2 t^2 - 5t + 4\,dt$

(b) Evaluate the integral from part (a).

Solution: $\int_0^2 t^2 - 5t + 4\,dt = \frac{t^3}{3} - \frac{5}{2}t^2 + 4t\Big|_0^2 = \frac{8}{3} - \frac{5}{2}(4) + 4(2) = \frac{8-6}{3} = \frac{2}{3}$

(c) Express the **distance** traveled by the particle over the interval $[0, 2]$ as a definite integral or as a sum of definite integrals. (Hint: Not the same answer as in (a).)

Solution: The distance traveled by a particle is the total amount traveled by the particle in either direction. To account for changes in direction, we can take the absolute value of the velocity function. We note that $v(t)$ changes from positive to negative on the interval $(1, 2)$. The distance traveled by the particle can thus be expressed as

$\int_0^2 |t^2 - 5t + 4|\,dt = \int_0^1 t^2 - 5t + 4\,dt + \int_1^2 -t^2 + 5t - 4\,dt$

(d) Evaluate the integral(s) from part (c).

Solution: $\int_0^1 t^2 - 5t + 4\,dt + \int_1^2 -t^2 + 5t - 4\,dt = \left(\frac{t^3}{3} - \frac{5}{2}t^2 + 4t\Big|_0^1\right) + \left(-\frac{t^3}{3} + \frac{5}{2}t^2 - 4t\Big|_1^2\right) =$
$\left(\frac{1}{3} - \frac{5}{2} + 4\right) + \left(-\frac{8}{3} + \frac{5}{2}(4) - 8 + \frac{1}{3} - \frac{5}{2} + 4\right) = \frac{1+1-8}{3} + \frac{5(4-1-1)}{2} + (4+4-8) = -2 + 5 = 3$

14.(2pts) You will be awarded these two points if your name appears in CAPITALS on the **front of your exam** and you mark your answers on the front page with an X through your answer choice like so: (a) (not an O around your answer choice) and you fill in the name of your instructor and your class time. You may also use this page for

ROUGH WORK